

Fractional Electromagnetic Waves

J.F. Gómez^a, J.J. Rosales^b, J.J. Bernal^a, V.I. Tkach^c, M. Guía^b

^aDepartamento de Física
División de Ciencias e Ingenierías Campus León
Universidad de Guanajuato
Lomas del Bosque s/n, Lomas del Campestre
León Guanajuato. México

^bDepartamento de Ingeniería Eléctrica
División de Ingenierías Campus Irapuato-Salamanca
Universidad de Guanajuato
Carretera Salamanca-Valle de Santiago, km. 3.5 + 1.8 km
Comunidad de Palo Blanco, Salamanca Guanajuato. México

^cDepartment of Physics and Astronomy
Northwestern University
Evanston, IL 60208-3112, USA

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Abstract: In the present work we consider the electromagnetic wave equation in terms of the fractional derivative of the Caputo type. The order of the derivative being considered is $0 < \gamma \leq 1$. A new parameter σ is introduced which characterizes the existence of the fractional components in the system. We analyze the fractional derivative with respect to time and space, for $\gamma = 1$ and $\gamma = 1/2$ cases.

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The recent interest on the fractional calculus (FC) and in particular in the fractional differential equations is stimulated by the applications in various areas of physics, chemistry, engineering and bioengineering [1]-[5]. Nevertheless, the derivation of such equations from some fundamental laws is not an easy matter. The fractional operator reflects intrinsic dissipative processes that are sufficiently complicated in nature. Their theoretical relationship with FC is not yet fully ascertained. Therefore, it is interesting to analyze a simple physical

system and try to understand their complete behavior given by the fractional differential equation.

In this work we will consider the electromagnetic wave equation in terms of the fractional derivative of the Caputo type. The solutions to the fractional differential wave equation are given in terms of the Mittag-Leffler function. First, we consider the fractional derivative with respect to time and second, the fractional derivative with respect to space.

The Maxwell equations for the electromagnetic waves in matter may be written as

$$\vec{\nabla} \cdot \vec{E} = \frac{4\pi}{\epsilon} \rho(\vec{r}, t), \quad (1)$$

$$\vec{\nabla} \cdot \vec{B} = 0, \quad (2)$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}, \quad (3)$$

$$\vec{\nabla} \times \vec{B} = \frac{4\pi\mu}{c} \vec{j}(\vec{r}, t) + \frac{\epsilon\mu}{c} \frac{\partial \vec{E}}{\partial t}, \quad (4)$$

where $\rho(\vec{r}, t)$ and $j(\vec{r}, t)$ are general, time-dependent distributions of charge densities and current densities, respectively. In (4) $\vec{D} = \epsilon \vec{E}$ is the dielectric displacement which is proportional to the electric field \vec{E} with the electric constant permittivity ϵ , and the magnetic field $\vec{B} = \mu \vec{H}$, where \vec{H} is the magnetic field intensity and μ the magnetic permeability. In the presence of matter the magnetic field intensity \vec{H} replaces the magnetic induction vector \vec{B} , in vacuum these field quantities are equal to each other $\vec{H} = \vec{B}$. In the case of homogeneous and isotropic medium the parameters ϵ and μ are constants, otherwise are vectors. Introducing the potentials, vector $\vec{A}(x_i, t)$ and scalar $\phi(x_i, t)$

$$\vec{B} = \vec{\nabla} \times \vec{A}, \quad (5)$$

$$\vec{E} = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t} - \vec{\nabla} \phi, \quad (6)$$

and using the Lorenz gauge condition we obtain the following decoupled differential equations for the potentials

$$\Delta \vec{A}(\vec{r}, t) - \frac{\epsilon\mu}{c^2} \frac{\partial^2 \vec{A}(\vec{r}, t)}{\partial t^2} = -\frac{4\pi}{c} \vec{j}(\vec{r}, t), \quad (7)$$

$$\Delta \phi(\vec{r}, t) - \frac{\epsilon\mu}{c^2} \frac{\partial^2 \phi(\vec{r}, t)}{\partial t^2} = -\frac{4\pi}{\epsilon} \rho(\vec{r}, t), \quad (8)$$

where $\frac{\epsilon\mu}{c^2} = \frac{1}{v^2}$. v is the velocity of the light in the medium.

The idea is to write the ordinary differential wave equations (1,2,3,4) and (7,8) in the fractional form with respect to t . For this, we propose to change

the ordinary time derivative operator by the fractional in the following way

$$\frac{d}{dt} \rightarrow \frac{1}{\sigma^{1-\gamma}} \frac{d^\gamma}{dt^\gamma}, \quad n-1 < \gamma \leq n, \quad (9)$$

where γ is an arbitrary parameter which represents the order of the time derivative, $0 < \gamma \leq 1$, and, σ , is a new parameter representing the fractional time components in the system, its dimensionality is the second. In the case $\gamma = 1$ the expression (9) transforms into ordinary time derivative operator

$$\frac{1}{\sigma^{1-\gamma}} \frac{d^\gamma}{dt^\gamma} \Big|_{\gamma=1} = \frac{d}{dt}. \quad (10)$$

The following Caputo definition of the fractional derivative will be used [1],

$$\begin{aligned} \frac{d^\gamma}{dt^\gamma} f(t) &= \frac{1}{\Gamma(n-\gamma)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\gamma+1-n}} d\tau, \\ n-1 < \gamma \leq n \in \mathbb{N} &= \{1, 2, \dots\}, \end{aligned} \quad (11)$$

where $\gamma \in \mathbb{R}$ is the order of the fractional derivative

$$f^{(n)}(\tau) = \frac{d^n f(\tau)}{d\tau^n}, \quad (12)$$

and

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \quad (13)$$

is the gamma function.

First, we consider the fractional time derivative. Then, using (10) the Maxwell equations (1-4) may be written in terms of the fractional time derivatives

$$\vec{\nabla} \cdot \vec{E} = \frac{4\pi\rho}{\epsilon}, \quad (14)$$

$$\vec{\nabla} \cdot \vec{B} = 0, \quad (15)$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{1}{\sigma^{1-\gamma}} \frac{\partial^\gamma \vec{B}}{\partial t^\gamma}, \quad (16)$$

$$\vec{\nabla} \times \vec{B} = \frac{4\pi\mu}{c} \vec{j} + \frac{\epsilon\mu}{c} \frac{1}{\sigma^{1-\gamma}} \frac{d^\gamma \vec{E}}{dt^\gamma}. \quad (17)$$

The relations (5,6) become

$$\vec{B} = \vec{\nabla} \times \vec{A}, \quad (18)$$

$$\vec{E} = -\frac{1}{c\sigma^{1-\gamma}} \frac{\partial^\gamma \vec{A}}{\partial t^\gamma} - \vec{\nabla}\phi. \quad (19)$$

Then, applying the Lorentz gauge condition we obtain the corresponding time fractional wave equations for the potentials (7,8)

$$\Delta \vec{A} - \frac{\epsilon\mu}{c^2} \frac{1}{\sigma^{2(1-\gamma)}} \frac{\partial^{2\gamma}}{\partial t^{2\gamma}} \vec{A} = -\frac{4\pi\mu}{c} \vec{j}, \quad (20)$$

$$\Delta \phi - \frac{\epsilon\mu}{c^2} \frac{1}{\sigma^{2(1-\gamma)}} \frac{\partial^{2\gamma}}{\partial t^{2\gamma}} \phi = -\frac{4\pi}{\epsilon} \rho. \quad (21)$$

In the case, $\gamma = 1$, the equations (20) and (21) become (7) and (8).

If, $\rho = 0$, and, $\vec{j} = 0$, we have the homogeneous fractional differential equations

$$\Delta \vec{A} - \frac{\epsilon\mu}{c^2} \frac{1}{\sigma^{2(1-\gamma)}} \frac{\partial^{2\gamma}}{\partial t^{2\gamma}} \vec{A} = 0, \quad (22)$$

$$\Delta \phi - \frac{\epsilon\mu}{c^2} \frac{1}{\sigma^{2(1-\gamma)}} \frac{\partial^{2\gamma}}{\partial t^{2\gamma}} \phi = 0. \quad (23)$$

We are interested in the analysis of the electromagnetic fields in the medium starting from the equations (22) and (23). We can write the fractional equations (22) and (23) in the following compact form

$$\frac{\partial^2 z(x, t)}{\partial x^2} - \frac{\epsilon\mu}{c^2} \frac{1}{\sigma^{2(1-\gamma)}} \frac{\partial^{2\gamma} z(x, t)}{\partial t^{2\gamma}} = 0, \quad (24)$$

where $z(x, t)$ represents both $\vec{A}(x, t)$ and $\phi(x, t)$. We consider a polarized electromagnetic wave, then $A_x(x_i, t) = 0$ $A_y(x_i, t) \neq 0$ $A_z(x_i, t) \neq 0$. The equation (24) is lineal and a particular solution may be found in the form

$$z(x, t) = z_0 e^{-ikx} \cdot u(t), \quad (25)$$

where k is the wavevector in the x direction and z_0 is a constant. Substituting (25) into (24), we obtain

$$\frac{\partial^{2\gamma} u(t)}{\partial t^{2\gamma}} + v^2 k^2 \sigma^{2(1-\gamma)} u(t) = 0. \quad (26)$$

Redefining

$$\omega^2 = v^2 k^2 \sigma^{2(1-\gamma)} = \omega_0^2 \sigma^{2(1-\gamma)}, \quad (27)$$

where ω_0 is the fundamental frequency of the electromagnetic wave, the equation (26) may be written as

$$\frac{\partial^{2\gamma} u(t)}{\partial t^{2\gamma}} + \omega^2 u(t) = 0. \quad (28)$$

The solution of this equation may be found in the form of the power series. The solution is

$$u(t) = E_{2\gamma} \left(-\omega^2 t^{2\gamma} \right), \quad (29)$$

where

$$E_{2\gamma}(-\omega^2 t^{2\gamma}) = \sum_{n=0}^{\infty} \frac{(-\omega^2 t^{2\gamma})^n}{\Gamma(2n\gamma + 1)}, \quad (30)$$

is the Mittag-Leffler function. Substituting the expression (29) in (25) we have a particular solution of the equation (24)

$$z(x, t) = z_0 e^{-ikx} \cdot E_{2\gamma}(-\omega^2 t^{2\gamma}). \quad (31)$$

In the first case, $\gamma = 1$, the Mittag-Leffler function (29) transforms into hyperbolic cosines and, from (27) $\omega = \omega_0$. Then

$$E_2(-\omega_0^2 t^2) = \cosh(\sqrt{-\omega_0^2 t^2}) = \cosh(i\omega_0 t) = \cos(\omega_0 t). \quad (32)$$

The expression (32) is a periodic function with respect to t . Therefore, in the case $\gamma = 1$, the solution to the equation (24) is

$$z(x, t) = \text{Re} z_0 e^{i(\omega_0 t - kx)}, \quad (33)$$

which defines a periodic, with fundamental period $T_0 = \frac{2\pi}{\omega_0}$, monochromatic wave in the, x , direction and in time, t . This result is very well known from the ordinary electromagnetic waves theory.

For the second case, $\gamma = 1/2$, the equation (24) becomes

$$\frac{\partial^2 z(x, t)}{\partial x^2} - \frac{\epsilon\mu}{c^2} \frac{1}{\sigma} \frac{\partial z(x, t)}{\partial t} = 0. \quad (34)$$

The solution may be found in the form of (25), then we obtain the following equation for the function $u(t)$

$$\frac{du}{dt} + \omega^2 u(t) = 0, \quad (35)$$

where, in this case, $\omega^2 = \omega_0^2 \sigma$, from (27). Solution of the equation (35) may be obtained in terms of the Mittag-Leffler function (29). In the case, $\gamma = 1/2$, we have

$$u(t) = E_1\{-\omega^2 t\} = e^{-\omega^2 t}. \quad (36)$$

The particular solution is

$$z(x, t) = z_0 e^{-\omega^2 t} e^{-ikx}, \quad (37)$$

For this case the solution is periodic only respect to x and it is not periodic with respect to t . The solution represents a plane wave with time decaying amplitude. The time in which the amplitude z_0 decay e times is

$$t_0 = \frac{1}{\omega^2} = \frac{1}{\omega_0^2 \sigma}. \quad (38)$$

It is important to note that, γ , is a dimensionless quantity which characterizes the order of fractional time derivative while the quantity, σ , has dimensions of time, and characterizes the presence of fractional time components in the medium. However, these two quantities are related as follows

$$\gamma = \sigma^2 \omega_0^2 = \frac{\sigma^2}{T_0^2} = \frac{\sigma_x^2}{\lambda^2}, \quad 0 < \sigma \leq T_0. \quad (39)$$

where, T_0 , is the period of the wave, λ , is the wavelength and, $\sigma = \frac{\sigma_x}{v}$, where, v , is the velocity of the electromagnetic wave in the, x , direction.

Taking into account this relation, the solution (29) may be written as

$$u(t) = E_{2\gamma} \left(-\gamma^{(1-\gamma)} \tilde{t}^{2\gamma} \right), \quad (40)$$

where, $\tilde{t} = \frac{t}{T_0}$, is a dimensionless parameter.

Now, we will consider the equation (24) assuming that the spatial derivative is fractional and the time derivative is ordinary. Then, we have the spatial fractional equation

$$\frac{1}{\sigma_x^{2(1-\delta)}} \frac{\partial^{2\delta} \tilde{z}(x, t)}{\partial x^{2\delta}} - \frac{1}{v^2} \frac{\partial^2 \tilde{z}(x, t)}{\partial t^2} = 0, \quad (41)$$

where the order of the fractional differential equation is represented by $0 < \delta \leq 1$, and σ_x has length dimension. A particular solution to the equation (41) may be as follows

$$\tilde{z}(x, t) = \tilde{z}_0 e^{i\omega t} u(x). \quad (42)$$

Substituting (42) in (41), we obtain

$$\frac{\partial^{2\delta} u(x)}{\partial x^{2\delta}} + \tilde{k}^2 u(x) = 0, \quad (43)$$

where

$$\tilde{k}^2 = \frac{\omega^2}{v^2} \sigma_x^{2(1-\delta)} = k^2 \sigma_x^{2(1-\delta)}, \quad (44)$$

is the wave-vector in the medium in presence of fractional components, and k is the wave vector in the medium without its presence. The wave-vectors are equal, $\tilde{k} = k$, only in the case, $\delta = 1$. Solution of the equation (43) is given in terms of the Mittag-Leffler function

$$u(x) = E_{2\delta}(-\tilde{k}^2 x^{2\delta}) = \sum_{n=0}^{\infty} \frac{(-\tilde{k}^2 x^{2\delta})^n}{\Gamma(2n\delta + 1)}. \quad (45)$$

First case: For the fractional spatial case, when $\delta = 1$, from (44) and (45), we have

$$E_2(-k^2 x^2) = \cosh(\sqrt{-k^2 x^2}) = \cosh(-ikx) = \text{Re}(e^{-ikx}). \quad (46)$$

In this case the solution follows from (42)

$$\tilde{z}(x, t) = \text{Re} \tilde{z}_0 e^{i\omega t - i k x}, \quad (47)$$

with, $\tilde{k} = k = \frac{\omega}{v}$, where, k , is the component of the wave-vector in the, x , direction and is related with the wavelength by, $k = \frac{1}{\lambda}$. The solution (47) represents a periodic, with respect to t and x , monochromatic wave.

Second case: For the case $\delta = 1/2$, we have from (44), $\tilde{k}^2 = k^2 \sigma_x = \frac{\omega^2}{v^2} \sigma$, and $[\tilde{k}^2] = \frac{1}{l}$ has dimensions of the inverse of the length. The solution for this case, has the form

$$u(x) = E_1 \left(-\tilde{k}^2 x \right) = e^{-\tilde{k}^2 x}. \quad (48)$$

The solution (42) is written as

$$\tilde{z}(x, t) = \tilde{z}_0 e^{i\omega t} e^{-\tilde{k}^2 x}. \quad (49)$$

The wave is periodic only with respect to t . The distance at which the amplitude \tilde{z}_0 is reduced e times is

$$x_0 = \frac{1}{\tilde{k}^2} = \frac{1}{k^2 \sigma_x}. \quad (50)$$

In this case we have that, δ , is a dimensionless quantity and σ_x is related to the fractional space. These two quantities are related by

$$\delta = k^2 \sigma_x^2 = \frac{\sigma_x^2}{\lambda^2}. \quad (51)$$

We can use this relation in order to write the equation (45) as follows

$$u(x) = E_{2\delta} \left(-\delta^{(1-\delta)} \tilde{x}^{2\delta} \right), \quad (52)$$

where, $\tilde{x} = \frac{x}{\lambda}$, is a dimensionless parameter. It can be seen that the solutions (40) and (52) have the same structure, and in the case, $\delta = \gamma$, have the same shape. Then, we can plot the function $u(s)$ where $s = (\tilde{x}, \tilde{t})$, for different values of the fractional parameter γ , (see Figure 1).

Conclusion: In this work we have studied the behavior of the electromagnetic waves applying the formalism of the fractional calculus. The order of the derivative being considered is $0 < \gamma \leq 1$. It showed that for the case where $\gamma = \delta = 1$ the solutions represent a periodic, with respect to t and x , monochromatic wave, as it should be. However, if we take $\gamma = 1/2$ the periodicity with respect to t is broken and behaves like a wave with time decaying amplitude Eq.(37). On the other hand, when $\delta = 1/2$ the periodicity with respect to x is broken and behaves like a wave with spatial decaying amplitude Eq.(49).

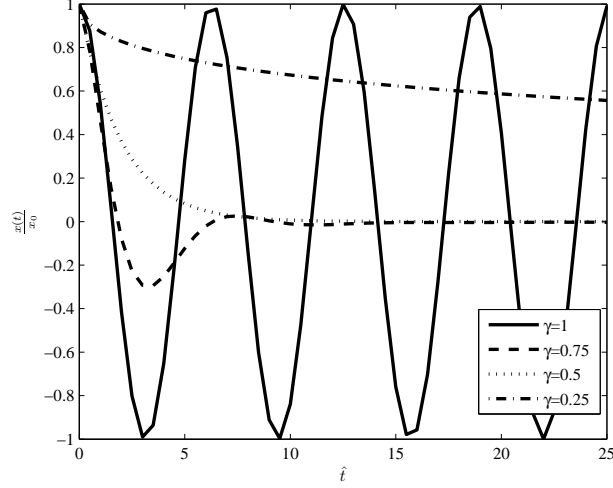


Figure 1: Graph corresponding to the equations (40) and (52)

References

- [1] I. Podlubny, *Fractional Differential Equations*, San Diego: Academic Press (1999); *The Laplace transform method for linear differential equations of the fractional order*. Tech. Rep., Slovak Academy of Sciences, Institute of Experimental Physics. (1994).
- [2] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Integrals and Derivatives of the Fractional Order and Some of Their Applications*, Nauka I Tekhnika, Minsk, (1987), (in Russian).
- [3] Shantanu Das, *Functional Fractional Calculus for System Identification and Controls*. Springer-Verlag Berlin Heidelberg (2008).
- [4] R.L. Magin, *Fractional Calculus in Bioengineering*, Connecticut: Begell House Publisher, (2006).
- [5] V.E. Tarasov, G.M. Zaslavsky, *Physica A* **354**, 249 (2005)